S.-T. Yau College Student Mathematics Contests 2025

Oral Exams in Geometry and Topology

Team (Solve 2 out of 3 problems)

1. Let Ω be the 2-form on $\mathbb{R}^3 - \{0\}$ defined by

$$\Omega = \frac{1}{(x^2+y^2+z^2)^{3/2}}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy).$$

Prove that Ω is not exact.

2. Let $\iota : L \to M$ be an immersion between compact topological manifolds L and M with dimension n and $2n$ respectively. Suppose the self intersection of the image $\iota(L)$ is transverse. Consider the fiber product

$$L \times_\iota L = \{(x,y) \in L \times L : \iota(x) = \iota(y)\}.$$

(1) Show that the self intersection must be a finite set of points.

(2) What are the connected components of $L \times_\iota L$?

(3) Now take L to be a disjoint union of k copies of two-dimensional spheres $\mathbb{S}^2$. We define the twisted Euler characteristic of the immersion to be

$$\chi(\iota(L)) := \chi(\Delta_L) - \sum_P \chi(P)$$

where $\Delta_L = \{(x,x) : x \in L\} \subset L \times_\iota L$ denotes the diagonal component, and P runs over all other components of $L \times_\iota L$, and χ denotes the Euler characteristic of the components. Show that $\chi(\iota(L))$ equals two times the Euler characteristic of the intersection graph Γ (as a curve), where Γ has k vertices corresponding to the sphere components and edges corresponding to intersection points between the sphere components.

3. Consider a smooth vector bundle E over a compact smooth manifold X with a connection ∇. Let $\Omega^\bullet(X,E)$ denote the space of smooth differential forms valued in E.

(1) What is the condition on ∇ such that $(\Omega^\bullet(X,E), \nabla)$ forms a chain complex?

(2) Given another connection $\tilde{\nabla}$ on E, show that it must be of the form $\nabla + A$ where A is a one-form valued endomorphism of E, that is, $A \in \Omega^1(X, End(E))$.

(3) Suppose ∇ satisfies the condition above. Consider the deformed chains $(\Omega^\bullet(X,E), \nabla + A_t)$ for a one-parameter smooth family of elements $A_t \in \Omega^1(X, End(E))$ with $A_0 = 0$. What is the condition on A_t such that it forms a chain complex? Show that this implies $\frac{dA_t}{dt}\big|_{t=0}$ is a closed element in degree one in the complex $(\Omega^\bullet(X, End(E)), \nabla)$.

(4) Could you elaborate on the geometric meaning of the first cohomology group of the complex $(\Omega^\bullet(X, End(E)), \nabla)$?